The Goldbach problem

This is a project for a student who likes problems about the distribution of prime numbers and who enjoyed the last part of the undergraduate course Analytic Number Theory related to the representation of every large enough positive integer as the sum of nine positive integer cubes.

One of the most well-known open problems in all of mathematics is the Goldbach conjecture: *every even integer greater than two is the sum of two primes.* If true, it implies that every odd integer greater than five is the sum of three primes, where repetition is allowed. These two statements are usually referred to as the binary and the ternary Goldbach problem respectively. While the binary problem is still unproved, Vinogradov managed to prove the ternary Goldbach problem for all large enough odd integers in 1937. His work proves the following stronger assertion: There exist absolute constants $N_0 > 5$ and $c_0 > 0$ such that every odd integer $N > N_0$ can be represented as a sum of 3 primes in at least

$$c_0 \frac{N^2}{(\log N)^3}$$

different ways. Goldbach's ternary problem was famously solved *for all* odd integers $5 < N \leq N_0$ by Harald Helfgott in 2013.

A leading component of Vinogradov's proof is the combination of the *prime number theorem for arithmetic progressions* and the *circle method*; both of these subjects were exposed to some extent (though in a rather different context) during the Analytic Number Theory undergraduate course.

This project is about studying the version of the proof given in the book of Davenport [1, Chapter 26]. The method is applicable to several problems regarding representations of integers by sums of primes; the student could for example decide whether there are specific subsets $\mathcal{P}$ of the primes such that each large enough positive integer in a special residue class is a sum of three primes all of which belong to $\mathcal{P}$.

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References

[1] Davenport, H. *Multiplicative number theory.* Graduate Texts in Mathematics, 74. *Springer-Verlag, New York*, 2000.